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# ON THE AVERAGE SHADOWING PROPERTY IN LINEAR DYNAMICAL SYSTEMS

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Abstract. In this paper we show that two notions of the average shadowing property and the hyperbolicity are equivalent in linear dynamical systems on the vector space $\mathbb{C}^n$.

## 1. Introduction

Let (X,d) be a metric space with a metric $d$ and $f : X \to X$ be a homeomorphism of the metric space $X$. As usual, we identify the homeomorphism $f$ with the dynamical system with discrete time generated by $f$ on $X$.

We give the main definitions in this paper for the most general case of dynamical system with discrete time generated by homeomorphisms: in fact, the main result of this paper is related to linear dynamical systems generated by nonsingular matrices (see [20]).

Fix a $\delta > 0$. A sequence $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is called a *$\delta$-pseudo orbit(or pseudotrajectory)* of $f$ if the following inequalities hold:

$$d(f(x_n), x_{n+1}) < \delta, \;\; n \in \mathbb{Z}.$$

We say that a dynamical system $f$ has the *shadowing property* if for every $\epsilon > 0$ we can find a $\delta > 0$ such that for any $\delta$-pseudo orbit $\xi = \{x_n : n \in \mathbb{Z}\}$ of $f$ there exists a point $y \in X$ such that

$$d(f^n(y), x_n) < \epsilon, \;\; n \in \mathbb{Z}.$$

The theory of shadowing of pseudo orbits in dynamical systems is now an important and rapidly developing branch of the modern global theory of dynamical systems. The notion of a pseudo orbit goes back to Birkhoff [3]. The real development of the shadowing theory started

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after the classical results of Anosov [1] and Bowen [6]. The main results obtained in recent years were reflected in the books [18, 20, 22]. It is known that every Axiom A diffeomorphism to the nonwandering set $\Omega$ has the shadowing property (see [6, Lemma]).

Blank introduced the notion of the average-shadowing property for studying properties of orbits of perturbed hyperbolic dynamical systems(see [4, 5]).

We recall that a sequence $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is called a $\delta$-*average pseudo orbit* of $f$ if there exists a positive integer $N = N(\delta)$ such that for all $n \geq N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that a dynamical system $f$ has the *average shadowing property* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every $\delta$-average pseudo orbit $\xi = \{x_n \in X : n \in \mathbb{Z}\}$ is $\epsilon$-shadowed on average by some $z \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

The notion of the average pseudo orbit is a generalization of the notion of the pseudo orbit. The notion of average shadowing property was further studied by several authors [5, 9, 12, 13, 14, 16, 17, 19, 23] with particular emphasis on connections with other notions known from topological dynamics, or more narrowly, shadowing theory.

When we study the average shadowing property in the qualitative theory of differentiable dynamical systems, an appropriate choice of the class of admissible average pseudo orbits is crucial here [9, 17, 23]. Moreover the average shadowing property is not related to the shadowing property in general.

Example 1.1. *Let $F : [0, 1] \to [0, 1]$ be a homeomorphism defined by*

$$F(t) = \begin{cases} t + (\frac{1}{2} - t)t & \text{if } \ 0 \leq t \leq \frac{1}{2} \\ t - (t - \frac{1}{2})(1 - t) & \text{if } \ \frac{1}{2} \leq t \leq 1. \end{cases}$$

*$F$ induces a homeomorphism $f : S^1 \to S^1$. Then $f$ has the shadowing property. But $f$ does have the average shadowing property* (*see* [19, Remark 3.3]).

*Kulczycki et al. give an example to illustrate that a map $f_1$ has the average shadowing property, but not the asymptotic average shadowing property* (*see* [10, Example 9.1]).

Let $\mathbb{C}^n$ be the vector space with a convenient vector norm $|\cdot| = d(\cdot,\cdot)$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ be a linear dynamical system with discrete time generated by a map $x \longmapsto Ax$ for a nonsingular matrix $A$.

We recall that a nonsingular matrix $A$ is called *hyperbolic* if its spectrum does not intersect the circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ (see [2]).

It is known that the following results hold for linear dynamical systems on $\mathbb{C}^n$:

1. A linear dynamical system $f$ generated by a nonsingular matrix $A$ has the shadowing property if and only if the matrix $A$ is hyperbolic (see [20]).
2. Various inverse shadowing properties and the hyperbolicity of a matrix $A$ are mutually equivalent (see [7]).
3. The limit shadowing property (ergodic shadowing property, respectively) and the hyperbolicity of matrix $A$ are equivalent in [11] ([15], respectively).

Also, it is known that if every homeomorphism $f$ satisfies the Smale Axiom A on the whole phase space $X$ then $f$ have the average shadowing property (see [4, Theorem 4]). But the converse does not hold in general.

In this paper we give an affirmative answer for the converse of Theorem 4 in [4], i.e., we show that if every linear dynamical system $f$ of the space $\mathbb{C}^n$ given by $f(x) = Ax$ has the average shadowing property, then the matrix $A$ is hyperbolic.

## 2. Main results

THEOREM 2.1. *For a linear dynamical system $f$ of $\mathbb{C}^n$ with discrete time generated by a map $x \longmapsto Ax$ for a nonsingular matrix $A$, the following two statements are equivalent:*

(a) *$f$ has the average shadowing property.*
(b) *The matrix $A$ is hyperbolic.*

For the proof of $(a) \Rightarrow (b)$, we need the following two lemmas.

LEMMA 2.2. *Assume that for two dynamical systems $f$ and $g$ of a metric space $X$ with a metric $d$, there exists a homeomorphism $h$ on $X$ such that $h$ and $h^{-1}$ are Lipschitz, and $g \circ h = h \circ f$. Then $f$ has the average shadowing property if and only if $g$ has the average shadowing property.*

*Proof.* Assume that $f$ has the average shadowing property. Let $\epsilon > 0$ be arbitrary and $L$ be a Lipschitz constant for $h$ and $h^{-1}$. Fix $\frac{\delta}{L} > 0$, let $\xi = \{x_n : n \in \mathbb{Z}\}$ be a $\frac{\delta}{L}$-average pseudo orbit of $f$. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \frac{\delta}{L} \text{ for all } k \in \mathbb{Z},$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \frac{\epsilon}{L}.$$

Since $h$ is Lipschitz and $g \circ h = h \circ f$, we get the following inequality:

$$\begin{aligned} \frac{1}{n}\sum_{i=0}^{n-1} d(g(h(x_{i+k})), h(x_{i+k+1})) &= \frac{1}{n}\sum_{i=0}^{n-1} d(h(f(x_{i+k})), h(x_{i+k+1})) \\ &\leq \frac{L}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta. \end{aligned}$$

So $\hat{\xi} = \{h(x_n) : n \in \mathbb{Z}\}$ is a $\delta$-average pseudo orbit of $g$. Also, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(g^i(h(z)), h(x_i)) &= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(h(f^i(z)), h(x_i)) \\ &\leq \limsup_{n\to\infty} \frac{L}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon. \end{aligned}$$

Thus the point $h(z) \in X$ is the $\epsilon$-average shadowed point of $g$. Hence $g$ has an average shadowing property. This completes the proof. □

LEMMA 2.3. [20] *Let $A$ be a nonhyperbolic matrix and $\lambda$ be an eigenvalue of $A$ with $|\lambda| = 1$. Then there exists a nonsingular matrix $T$ such that $J = T^{-1}AT$ is a Jordan form of $A$ and the matrix $J$ has the form $\begin{pmatrix} B & O \\ O & D \end{pmatrix}$ where $B$ is the nonsingular $m \times m$ complex matrix with the form*

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

*and $D$ is the hyperbolic matrix.*

*Proof of* $(a) \Rightarrow (b)$. Suppose that $f$ has the average shadowing property. To derive a contradiction, we may assume that the matrix $A$ is nonhyperbolic. Then the matrix $A$ has an eigenvalue $\lambda$ with $|\lambda| = 1$. By Lemma 2.3, there is a nonsingular matrix $T$ such that $J = T^{-1}AT$ is a Jordan form of $A$ and the Jordan form $J = \begin{pmatrix} B & O \\ O & D \end{pmatrix}$, where $B$ and $D$ are as in Lemma 2.3. Let $g(x) = J(x) = T^{-1}AT(x)$, and let $h(x) = T(x)$ for $x \in \mathbb{C}^n$. Then $f \circ h = h \circ g$. From the average shadowing property of $f$ and Lemma 2.2, $g$ also has the average shadowing property. Let $\delta > 0$ be the number of the definition of the average shadowing property of $g$. Denote by $x^{(i)}$ the $i$-th component of a vector $x \in \mathbb{C}^n$. Then we construct a $\delta$-pseudo orbit as follows:

$$x^{(1)}_{i+k+1} = \lambda x^{(1)}_{i+k}\Big(1 + \frac{\delta}{2|x^{(1)}_{i+k}|}\Big), \quad \text{and}$$

$$\begin{aligned} x'_{i+k+1} &= \Big(x^{(2)}_{i+k+1}, x^{(3)}_{i+k+1}, \ldots, x^{(n)}_{i+k+1}\Big) \\ &= \Big((Jx_{i+k})^{(2)}, (Jx_{i+k})^{(3)}, \ldots, (Jx_{i+k})^{(n)}\Big) \quad \text{for all } i, k \in \mathbb{Z}. \end{aligned}$$

Since $g(x_{i+k}) = Jx_{i+k} = \Big(\lambda x^{(1)}_{i+k}, (Jx_{i+k})^{(2)}, (Jx_{i+k})^{(3)}, \ldots, (Jx_{i+k})^{(n)}\Big) = \Big(\lambda x^{(1)}_{i+k}, x'_{i+k+1}\Big)$, we know that if $\lambda = 1$, then

$$d(g(x_{i+k}), x_{i+k+1}) = \Big|x^{(1)}_{i+k} - x^{(1)}_{i+k} - \frac{x^{(1)}_{i+k}\delta}{2|x^{(1)}_{i+k}|}\Big| = \frac{\delta}{2} < \delta,$$

for all $i, k \in \mathbb{Z}$. Thus $\xi = \{x_n : n \in \mathbb{Z}\}$ is a $\delta$-pseudo orbit of $g$. Also, this sequence is a $\delta$-average pseudo orbit of $g$.

Since $g$ has the average shadowing property, there exists a point $z \in \mathbb{C}^n$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(g^i(z), x_i) < \epsilon.$$

Now, we will consider the following two cases.

First, let $z = (0, 0, \ldots, 0)$, then $d(g^i(z), x_i) = |x_i| > 0$, which leads us to the following inequality:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(g^i(z), x_i) > |x_j|,$$

where $|x_j| = \min\{|x_i| : \text{ for all } i \in \mathbb{Z}\}$. This is a contradiction.

Second, if $z = \left(z^{(1)}, z^{(2)}, z^{(3)}, \ldots, z^{(n)}\right)$, then

$$g^i(z) = \left(z^{(1)}, (J^i z)^{(2)}, (J^i z)^{(3)}, \ldots, (J^i z)^{(n)}\right)$$

and $d(g^i(z), x_i) \geq ||g^i(z)| - |x_i||$. Using this formula we get the following inequality:

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(g^i(z), x_i) \geq ||g^k(z)| - |x_k||,$$

where $||g^k(z)| - |x_k|| = \min\{||g^i(z)| - |x_i|| : \text{for all } i \in \mathbb{Z}\}$.
Here, we take $\epsilon_2 = \frac{\min\{||g^i(z)|-|x_i||\}}{2}$, then we get a contradiction.
Therefore, if we choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, where $\epsilon_1 = \frac{\min\{|x_i|\}}{2}$, we can not find $z$ that is $\epsilon$-average shadowed. Thus if $f$ has the average shadowing property, then the matrix $A$ is hyperbolic. □

*Proof of* $(b) \Rightarrow (a)$. Finally, we show that $(b) \Rightarrow (a)$. We will introduce the following definition by [8] for the proof. Easton [8] introduced the following property of a dynamical system $f$ on a metric spaces $(X, d)$: given $\epsilon > 0$ there exists $\delta > 0$ such that if a sequence $\xi = \{x_n : n \in \mathbb{Z}\}$ satisfies the inequality

$$\sum_{n\in\mathbb{Z}} d(f(x_n), x_{n+1},) < \delta,$$

then there exists a point $y$ such that

$$\sum_{n\in\mathbb{Z}} d(f^n(y), x_n) < \epsilon.$$

This is called *strong shadowing property*. Pilyugin showed that if $A$ is hyperbolic then $f$ has the strong shadowing property (see [20]).

Lemma 2.4. *Let* $f : \mathbb{C}^n \to \mathbb{C}^n$ *be as before. If* $f$ *has the strong shadowing property, then it has the average shadowing property.*

*Proof.* Suppose that $f$ has the strong shadowing property. For any $\delta > 0$, let $\xi = \{x_n : n \in \mathbb{Z}\}$ be a $\delta$-strong pseudo orbit of $f$, that is,

$$\sum_{n\in\mathbb{Z}} d(f(x_n), x_{n+1}) = \sum_{n\in\mathbb{Z}} |A(x_n) - x_{n+1}| < \delta.$$

It is clear that the $\delta$-strong pseudo orbit $\xi = \{x_n : n \in \mathbb{Z}\}$ is a $\delta$-average pseudo orbit of $f$ since

$$\frac{1}{n}\sum_{i=0}^{n-1}|A(x_{i+k}) - x_{i+k+1}| < \sum_{n\in\mathbb{Z}} d(f(x_n), x_{n+1}) = \sum_{n\in\mathbb{Z}}|A(x_n) - x_{n+1}| < \delta.$$

Since $f$ has the strong shadowing property, there exists a point $y \in \mathbb{C}^n$ such that $\sum_{n\in\mathbb{Z}} d(f^n(y), x_n) < \epsilon$. It follows from this formula that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(y), x_i) < \sum_{n\in\mathbb{Z}} d(f^n(y), x_n) < \epsilon.$$

It means that $y$ is an $\epsilon$-shadowed point in $\mathbb{C}^n$. The proof is complete. □

LEMMA 2.5. *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be as before. If the nonsingular matrix $A$ is hyperbolic, then $f$ has the average shadowing property.*

*Proof.* Assume that the matrix $A$ is hyperbolic. Then $f$ has the strong shadowing property by [20, Remark, p.70]. Therefore $f$ has the average shadowing property by Lemma 2.4. □

It is well known that if $f$ has the asymptotic average shadowing property, then it is average shadowing property. Furthermore, it is proved that the various shadowing properties and hyperbolicity are equivalent by [7, 11, 14, 15, 20, 21], i.e., we obtain the following results for linear dynamical systems of $\mathbb{C}^n$.

REMARK 2.6. Given a linear dynamical system $f$ of $\mathbb{C}^n$ generated by a matrix $A$, we have the following results:
A linear dynamical system $f$ has the shadowing (respectively orbital shadowing, limit shadowing, average shadowing, asymptotic average shadowing, ergodic shadowing, inverse shadowing, weak inverse, orbital inverse) property if and only if the matrix $A$ is hyperbolic.

We can complete the proof of $(b) \Rightarrow (a)$ of Theorem 2.1 by using the above properties. □

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## References

[1] D. V. Anosov, *On a class of invariant sets of smooth dynamical systems*, In: Proc. 5th Int. Conf. Nonl. Oscill. Kiev, **2** (1970), 39-45

[2] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems. Recent Advances*, North-Holland Math. Library, **52**, North-Holland, Amsterdam, 1994.

[3] G. Birkhoff, *An extension of Poincaré's last geometric theorem*, Acta Math. **47** (1926), 297-311.

[4] M. L. Blank, *Metric properties of $\varepsilon$-trajectories of dynamical systems with stochastic behaviour*, Ergod. Th. & Dynam. Sys. **8** (1988), 365-378.

[5] M. L. Blank, *Small perturbations of chaotic dynamical systems*(Russian), Uspekhi Mat. Nauk, **44** (1989), no. 6(270), 3-28, 203; translation in Russian Math. Surveys, **44** (1989), no. 6, 1-33.

[6] R. Bowen, *$\omega$-Limit sets for Axiom A diffeomorphisms*, J. Differential Equations, **18** (1975), 333-339.

[7] T. Choi and K. Lee, *Various inverse shadowing in linear dynamical systems*, Commun. Korean Math. Soc. **21** (2006), no. 3, 515–526.

[8] R. Easton,*Chain transitivity and domain of influence of an invariant set*, The Structure of Atractors in Dynamical Systems (North Dacota State Univ. June 1977), Lecture Notes in Math. **668** (Springer Verlag, Berlin, 1978), 95-102.

[9] K. Kulczycki, D. Kwietniak, and P. Oprocha, *On almost specification and average shadowing properties*, Fund. Math. **224** (2014), no. 3, 241-278.

[10] D. Kwietniak and P. Oprocha, *A note on the average shadowing property for expansive maps*, Topology Appl. **159** (2012), no. 1, 19-27.

[11] K. Lee, M. Lee, and J. Park, *Linear diffeomorphisms with limit shadowing*, J. Chungcheong Math. Soc. **26** (2013), 309-313.

[12] M. Lee, *Average shadowing property on closed sets*, Far East J. Math. Sci. **57** (2011), 171-179.

[13] M. Lee, *Stably average shadowable homoclinic classes*, Nonliear Anal. **74** (2011), 689-694.

[14] M. Lee, *Asymptotic average shadowing in linear dynamical systems,* Far East J. Math. Sci. **66** (2012), 37-44.

[15] M. Lee, *Linear dynamical systems with ergodic shadowing,* Far East J. Math. Sci. **68** (2012), 239-244.

[16] M. Lee and X. Wen, *Diffeomorphisms with $C^1$-stably average shadowing*, Acta Math. Sin. Eng. Seri. **29** (2013), 85–92.

[17] Y. X. Niu, *The average-shadowing property and strong ergodicity*, J. Math. Anal. Appl. **376** (2011), 528-534.

[18] K. Palmer, *Shadowing in Dynamical Systems: Theory and Applications*, Kluwer, Dordrecht, 2000.
[19] J. Park and Y. Zhang, *Average shadowing properties on compact metric spaces*, Commun. Korean Math. Soc. **21** (2006), no. 2, 355-361.
[20] S. Y. Pilyugin, *Shadowing in Dynamical Systems*, Lecture Notes in Math. **1706**, Springer Verlag, Berlin, 1999.
[21] S. Y. Pilyugin, A. A. Rodionova, and K. Sakai, *Orbital and weak shadowing properties*, Discrete Contin. Dynam. Syst. **9** (2003), 287-308.
[22] S. Y. Pilyugin and K. Sakai, *Shadowing and Hyperbolicity*, Lecture Notes in Math. **2193**, Springer International Publishing AG, 2017.
[23] Y. Zhang, *On the average-shadowing property*, Acta Sci. Natur. Univ. Pekinensis, **37** (2001), 648-651.

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